[5] Petrick, E. N. and Krauss, J. J., "The design and evaluation of perforated ion emitters" Final TR, Contract AF 33(657)-8638, performed for Aeronaut. Systems Div., U. S. Air Force, by Kelsey Hayes Co., Romulus, Mich. (September 15, 1963).

[6] Winterbottom, W. L. and Hirth, J. P., "Diffusional contribution to the total flow from a knudsen cell," J. Chem. Phys. **37**, 784–793 (1962).

Thermal Boundary Layer in Slip Flow Regime

K. C. Reddy*

Indian Institute of Technology, Kharagpur, India

Velocity-slip and temperature-jump boundary conditions replacing the classical conditions of no-slip and continuous temperature distribution have formed the basis of study of many a problem[1–5] in rarefied gas dynamics. First-order slip boundary conditions (neglecting thermal creep terms) are given by

$$u(x,0) = L(\partial u/\partial y)_{y=0} \tag{1.1}$$

$$T(x,0) - T_w = L_1(\partial T/\partial y)_{y=0} \tag{1.2}$$

where

$$L = \frac{2-f}{f}\lambda \qquad L_1 = \frac{2-f_1}{f_1}\frac{2\gamma}{\gamma+1}\frac{\lambda}{Pr}$$

f being the Maxwell's reflection coefficient, f_1 the thermal accommodation coefficient, λ the mean free path of the fluid at the surface, γ the specific heat ratio, Pr the Prandtl number, and T_w the wall temperature.

Putting

$$T - T_w = u^{L/L_1}\theta = u^\alpha\theta \tag{2}$$

where $\alpha = L/L_1$, the condition (1.2) reduces, on using (1.1), to the form

$$(\partial\theta/\partial y)_{y=0} = 0 \tag{3}$$

Thus the transformation (2) simplifies the temperature jump condition considerably. It is much easier to manage with condition (3) than its original form (1.2), particularly when Von Mises' transformation is used.

Hasimoto[2] has obtained a solution in power series of x to the boundary-layer momentum equation for the flow past a flat plate under slip conditions using Von Mises' transformation. Hassan[3] has extended this analysis to include the flows of a certain class of outer pressure distributions. In the present note, the author will follow Hassan's analysis to solve the corresponding energy equation using the transformation (2) and thus the simplified boundary condition (3). Two-dimensional incompressible boundary-layer momentum and energy equations are given by

$$uu_x + vu_y = VV_x + \nu u_{yy} \tag{4.1}$$

$$u_x + v_y = 0 \tag{4.2}$$

$$\rho c_p(uT_x + vT_y) = kT_{yy} + \mu(u_y)^2 \tag{4.3}$$

where V is the freestream velocity. Introducing the transformation (2), Eq. (4.3) becomes

$$\rho c_p\{u(u^\alpha\theta)_x + v(u^\alpha\theta)_y\} = k(u^\alpha\theta)_{yy} + \mu(u_y)^2 \tag{5}$$

The function θ should satisfy the condition (3) at the wall, and

$$\theta \to \frac{T_\infty - T_w}{V^\alpha} \quad \text{as } y \to \infty \tag{6}$$

where T_∞ is the freestream temperature. Taking (x,ψ), where ψ is the stream function, as independent variables instead of (x,y), Eq. (5) can be written as

$$(u^\alpha\theta)_x = \frac{\nu}{Pr}\{u(u^\alpha\theta)_\psi\}_\psi + \frac{\nu}{c_p}u(u_\psi)^2 \tag{7}$$

and the boundary conditions (3) and (6) reduce to

$$\theta_\psi = 0 \text{ at } \psi = 0 \qquad \theta \to \frac{T_\infty - T_w}{V^\alpha} \text{ as } \psi \to \infty \tag{8}$$

Introducing the dimensionless variables defined by

$$V = \frac{\nu}{L}h(\xi) \qquad \xi = \left(\frac{x}{L}\right)^{1/2} \qquad \eta = \frac{\psi}{\nu h}\left(\frac{L}{x}\right)^{1/2}$$
$$u = \frac{\nu}{L}h(\xi)\phi(\xi,\eta) \qquad \theta = \left(\frac{\nu}{L}\right)^{2-\alpha}\frac{g(\xi,\eta)}{c_p h^\alpha} \tag{9}$$

Eq. (7) takes the form

$$h\xi\frac{\partial}{\partial\xi}(\phi^\alpha g) - \left(h + \xi\frac{dh}{d\xi}\right)\eta\frac{\partial}{\partial\eta}(\phi^\alpha g) = \frac{2}{Pr}\frac{\partial}{\partial\eta}\left\{\phi\frac{\partial}{\partial\eta}(g\phi^\alpha)\right\} + 2h^2\phi\left(\frac{\partial\phi}{\partial\eta}\right)^2 \tag{10}$$

Correspondingly, the boundary conditions (8) are transformed to

$$\partial g/\partial\eta = 0 \text{ at } \eta = 0 \qquad g \to g_\infty \text{ as } \eta \to \infty \tag{11}$$

where

$$g_\infty = c_p(T_\infty - T_w)(L/\nu)^2$$

Following Hassan, assume series solutions for ϕ and g in powers of ξ, viz.,

$$\phi = 1 + \sum_{n=1}^{\infty}\phi_n(\eta)\xi^n \qquad g = g_\infty + \sum_{n=1}^{\infty}g_n(\eta)\xi^n \tag{12}$$

and suppose that h can be expanded into the form

$$h = \sum_{n=0}^{\infty}a_n\xi^n \qquad a_0 \neq 0 \tag{13}$$

Now consider the cases for $\alpha = 1$ and $\alpha = \frac{1}{2}$. It may be remarked that experimental and theoretical evidence[4] corresponds more to the latter case.

The Case of $\alpha = 1$

Substituting (12) and (13) into (10) and equating the coefficients of like powers of ξ, one obtains the equations for g_n as

$$g_n'' + \frac{Pr\,a_0}{2}(\eta g_n' - ng_n) = S_n(\eta) \tag{14}$$

where $S_n(\eta)$ involves $g_1, g_2, \ldots g_{n-1}, \phi_1, \ldots \phi_n$, their derivatives, and the constants $a_0, a_1, \ldots a_{n-1}$. Introducing the variables

$$G_n = \frac{Pr\,a_0}{4}g_n \qquad \zeta = \frac{(Pr\,a_0)^{1/2}}{2}\eta$$

Eq. (14) can be written as

$$G_n'' + 2(\zeta G_n' - nG_n) = S_n(\zeta) \tag{15}$$

where primes denote differentiation with respect to ζ.

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* CSIR Junior Research Fellow, Department of Mathematics.

Boundary conditions for G_n are obtained from (11):

$$G_n' = 0 \text{ at } \zeta = 0 \qquad G_n \to 0 \text{ as } \zeta \to \infty \tag{16}$$

Similarly, introducing

$$F_n = (a_0/4)\phi_n \qquad z = (a_0^{1/2}/2)\eta$$

Hassan has obtained the differential equation for F_n as

$$F_n'' + 2(zF_n' - nF_n) = R_{n-1}(z) \tag{17}$$

He has solved these equations for F_1, F_2, and F_3 in terms of repeated integrals of complementary error functions and has calculated the skin friction coefficient. One can, however, express these functions of F_n in terms of error functions alone instead of their integrals. For example,

$$F_1(z) = \frac{a_0^{1/2}}{2}\left\{\frac{1}{\pi^{1/2}}\exp(-z^2) - z + zE(z)\right\}$$

$$F_2(z) = \frac{1}{2}\left\{\frac{1}{4}\left[(1+2z^2)(1-E(z)) - \frac{2}{\pi^{1/2}}z\exp(-z^2)\right]\times \left[\frac{5}{2} + E(z) + a_1\left(\frac{\pi}{a_0}\right)^{1/2}\right] - \left[\frac{1}{\pi^{1/2}}\exp(-z^2) - z + zE(z)\right]\times \left[\frac{a_1}{a_0^{1/2}} + \frac{1}{\pi^{1/2}}\exp(-z^2) - z + zE(z)\right] - \frac{2a_1}{3(\pi a_0)^{1/2}}\exp(-z^2)\right\} \tag{18†}$$

where

$$E(z) = \frac{2}{(\pi)^{1/2}}\int_0^z \exp(-t^2)\,dt$$

Using the solution for F_1, S_1 of Eq. (15) is found to be

$$S_1(\zeta) = g_\infty\left(\frac{a_0}{\pi}\right)^{1/2}(Pr-1)\exp\left(-\frac{\zeta^2}{Pr}\right) \tag{19}$$

For the case of $Pr = 1$, $S_1(\zeta)$ vanishes and one gets

$$G_1(\zeta) = 0 \tag{20}$$

In this case of $Pr = 1$, $S_2(\zeta)$ reduces, on some simplification, to

$$S_2(\zeta) = -a_0^2\{E(\zeta) - 1\}^2 - \frac{2a_1g_\infty}{a_0^{1/2}}\times \left\{\zeta E(\zeta) - \zeta + \frac{\exp(-\zeta^2)}{\pi^{1/2}}\right\} \tag{21}$$

The solution for G_2 is found to be

$$G_2(\zeta) = \frac{g_\infty a_1}{a_0^{1/2}}\left\{[E(\zeta)-1]\left[\frac{\pi^{1/2}}{2}\zeta^2 + \zeta + \frac{\pi^{1/2}}{4}\right] + \exp(-\zeta^2)\left[\frac{\zeta}{2} + \frac{1}{\pi^{1/2}}\right]\right\} + \frac{a_0^2}{2}\left\{E(\zeta)\left[-\zeta^2E(\zeta) + \zeta^2 - \frac{2\zeta\exp(-\zeta^2)}{\pi^{1/2}} - \frac{1}{2}\right] - \frac{\exp(-2\zeta^2)}{\pi} + \frac{\zeta\exp(-\zeta^2)}{\pi^{1/2}} + \frac{1}{2}\right\} \tag{22}$$

When $Pr \neq 1$, the solution for G_1 is found to be

$$G_1(\zeta) = \frac{g_\infty(a_0Pr)^{1/2}}{2}\left\{\zeta\left[E(\zeta) - E\left(\frac{\zeta}{Pr^{1/2}}\right)\right] + \frac{\exp(-\zeta^2)}{\pi^{1/2}} - \left(\frac{Pr}{\pi}\right)^{1/2}\exp\left(\frac{-\zeta^2}{Pr}\right)\right\} \tag{23}$$

The solution for G_2 becomes too cumbersome and is not given here.

† Incidentally, Eq. (22a) in Hassan's paper[3] should have $-a_1\beta/4$ instead of $+a_1\beta/4$ in the second term of the expression for F_2. Correspondingly, his Eqs. (22b) and (23) should be modified.

The case of $\alpha = \frac{1}{2}$

In this case, Eq. (10) contains terms involving $\phi^{1/2}$. Therefore, retain the series assumption for ϕ, g, and h as in (12) and (13) and further assume that

$$\phi^{1/2} = 1 + \sum_{n=1}^{\infty} P_n(\eta)\xi^n \tag{24}$$

Squaring both sides of (24) and comparing the coefficients of various powers of ξ, after substituting the expression for ϕ from (12), one gets

$$P_1(\eta) = \phi_{1/2} \qquad P_2(\eta) = \frac{\phi_2}{2} - \frac{\phi_1^2}{8} \qquad \ldots \tag{25}$$

When the series for $\phi^{1/2}$ given by (24) and (25) is put into (10), one gets equations similar to (14) with $S_n(\eta)$ standing for slightly different expressions. Again, for $Pr = 1$, one gets

$$G_1(\zeta) = 0 \tag{26}$$

and

$$G_2(\zeta) = \frac{g_\infty a_1}{2a_0^{1/2}}\left\{[E(\zeta)-1]\left[\frac{\pi^{1/2}}{2}\zeta^2 + \zeta + \frac{\pi^{1/2}}{4}\right] + \exp(-\zeta^2)\left(\frac{\zeta}{2} + \frac{1}{\pi^{1/2}}\right)\right\} + \frac{1}{2}\left(a_0^2 - \frac{g_0}{4}\right)\times \left\{E(\zeta)\left[-\zeta^2E(\zeta) - \frac{2\zeta}{\pi^{1/2}}\exp(-\zeta^2) + \zeta^2 - \frac{1}{2}\right] - \frac{\exp(-2\zeta^2)}{\pi} + \frac{\zeta\exp(-\zeta^2)}{\pi^{1/2}} + \frac{1}{2}\right\} \tag{27}$$

And for $Pr \neq 1$,

$$G_1(\zeta) = \frac{g_\infty(a_0Pr)^{1/2}}{4}\left\{\zeta\left[E(\zeta) - E\left(\frac{\zeta}{Pr^{1/2}}\right)\right] + \frac{\exp(-\zeta^2)}{\pi^{1/2}} - \left(\frac{Pr}{\pi}\right)^{1/2}\exp\left(\frac{-\zeta^2}{Pr}\right)\right\} \tag{28}$$

Heat Transfer Rate

Heat transfer rate at the wall is given by

$$q = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} = -\frac{k}{L_1}[(T)_{y=0} - T_w] = -\frac{k}{L_1}(u^\alpha\theta)_{y=0}$$
$$= -\frac{k}{L_1}\left[\frac{LV^2}{2\nu}c_f\right]^\alpha(\theta)_{y=0} \tag{29}$$
$$= -\frac{k}{L_1c_p}\left(\frac{\nu}{L}\right)^2\left[\frac{c_fh}{2}\right]^\alpha(g)_{y=0}$$

where c_f is the local skin friction coefficient calculated by Hassan.

For $\alpha = 1$,

$$q = -\frac{k\nu^2c_fh}{2c_pL^3}\left[g_\infty + \frac{2g_\infty(1-Pr^{1/2})}{(a_0\pi Pr)^{1/2}}\xi + \ldots\right] \qquad Pr \neq 1$$
$$= -\frac{k\nu^2c_fh}{2c_pL^3}\left\{g_\infty + \frac{4}{a_0}\left[\frac{a_0^2}{2}\left(-\frac{1}{\pi} + \frac{1}{2}\right) + \frac{g_\infty a_1}{a_0^{1/2}}\left(\frac{1}{\pi^{1/2}} - \frac{\pi^{1/2}}{4}\right)\right]\xi^2 + \ldots\right\} \qquad Pr = 1 \tag{30}$$

For $\alpha = \frac{1}{2}$,

$$q = -\frac{k\nu^2}{2c_pL^3}\left(\frac{c_fh}{2}\right)^{1/2}\left[g_\infty + \frac{g_\infty(1-Pr^{1/2})}{(a_0\pi Pr)^{1/2}}\xi + \ldots\right] \qquad Pr \neq 1$$

$$= -\frac{k\nu^2}{2c_pL^3}\left(\frac{c_fh}{2}\right)^{1/2}\left\{g_\infty + \frac{4}{a_0}\left[\frac{1}{2}\left(a_0^2 - \frac{g_0}{4}\right) \times \left(-\frac{1}{\pi} + \frac{1}{2}\right) + \frac{g_\infty a_1}{2a_0^{1/2}}\left(\frac{1}{\pi^{1/2}} - \frac{\pi^{1/2}}{4}\right)\right]\xi^2 + \ldots\right\}$$

$$Pr = 1 \quad (31)$$

References

[1] Schaaf, S. A. and Sherman, F. S., "Skin friction in slip flow," J. Aeronaut. Sci. 21, 85–90, 144 (1954).

[2] Hasimoto, H., "Boundary layer slip solutions for a flat plate," J. Aeronaut. Sci. 25, 68–69 (1958).

[3] Hassan, H. A., "On skin friction in the slip-flow regime," J. Aeronaut. Sci. 28, 335–336 (1961).

[4] Street, R. E., "A study of boundary conditions in slip-flow aerodynamics," *Rarefied Gas Dynamics* (Pergamon Press, Inc., New York, 1960), pp. 276–292.

[5] Sparrow, E. M. and Lin, S. H., "Laminar heat transfer in tubes under slip-flow conditions," Trans. Am. Soc. Mech. Engrs. Ser. C: J. Heat Transfer 84, 363–369 (November 1962).

Pressure Distribution for Hypersonic Boundary-Layer Flow

M. S. Sastry*

Indian Institute of Technology, Kanpur, India

A pressure distribution is derived here, which is in excellent agreement with the tangent-wedge approximation.

Introduction

In solving the hypersonic boundary-layer momentum integral equation [Eq. (2.12) of Ref. 1], it is necessary to find out the pressure distribution $P^*(= P_2/P_1$, where subscript 2 stands for $x = x$, $y = \delta$, and subscript 1 stands for $x = \infty$, $y = \delta$; x, y are the surface and normal-to-surface coordinates, and δ is the boundary-layer thickness) in terms of η and ξ, where $\eta = \delta/L$ and $\xi = x/L$ (where L equals a certain characteristic length defined in Ref. 1). Here, one such pressure distribution is derived from the fundamental shock-relations.

Shock Relations and Derivation of Pressure Distribution

From shock relations,

$$P_2/P_{-\infty} = [2\gamma/(\gamma + 1)]M_{-\infty}^2 \sin^2\theta - [(\gamma - 1)/(\gamma + 1)] \quad (1)$$

where γ is the specific heats ratio; $M_{-\infty}$ is the freestream Mach number; θ is the shock angle; and $P_{-\infty}$ is the freestream pressure.

Equation (1) also can be written as

$$(P_2 - P_{-\infty})/P_{-\infty} = (2\gamma/\gamma + 1)(M_{-\infty}^2 \sin^2\theta - 1) \quad (2)$$

Also, shock angle and deflection angle relation is

$$M_{-\infty}^2 \sin^2\theta - 1 = \frac{\gamma + 1}{2}\frac{M_{-\infty}^2 \sin\theta \sin\Delta}{\cos(\theta - \Delta)} \quad (3)$$

where Δ is the deflection angle.

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* Junior Research Fellow, Department of Mathematics.

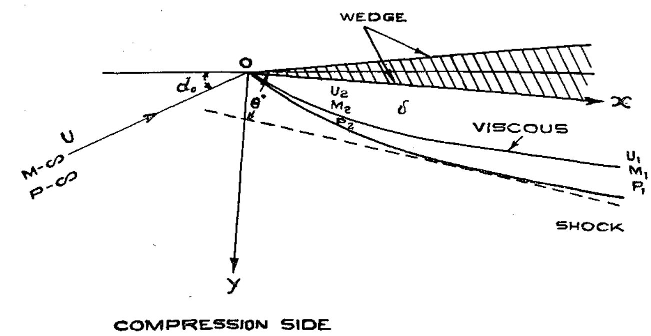

Fig. 1 Hypersonic viscous flow over an insulated wedge at an angle of attack.

Now for slender wedges, $\cos(\theta - \Delta) \approx 1$ and $\sin\Delta \approx \Delta$. Therefore (3) becomes

$$M_{-\infty}^2 \sin^2\theta - 1 = [(\gamma + 1)/2]M_{-\infty}^2 \sin\theta \cdot \Delta \quad (4)$$

Hence,

$$\frac{P_2 - P_{-\infty}}{P_{-\infty}} = \frac{2\gamma}{\gamma + 1}\left[\frac{\gamma + 1}{2} M_{-\infty}^2 \sin\theta\right]\Delta$$

$$= \gamma M_{-\infty}^2 \sin\theta \cdot \Delta \quad (5)$$

Now as in Ref. 1 in order to satisfy the asymptotic conditions at $x = +\infty$, let $\theta = \theta_0 + \theta_1$ and $\Delta = \Delta_0 + \Delta_1$, where θ_0 and Δ_0 are the inviscid shock and deflection angles, and θ_1 and Δ_1 correspond to viscous shock and deflection angles. So, (5) becomes

$$P_2/P_{-\infty} = 1 + \{\gamma M_{-\infty}^2 \sin(\theta_0 + \theta_1)\}(\Delta_0 + \Delta_1)$$
$$= 1 + \gamma M_{-\infty}^2(\Delta_0 + \Delta_1)(\sin\theta_0 + \theta_1 \cos\theta_0)$$

where $\cos\theta_1 \approx 1$ and $\sin\theta_1 \approx \theta_1$. Therefore,

$$P_2/P_{-\infty} = 1 + \gamma M_{-\infty}^2 \sin\theta_0 \cdot \Delta_0 + \gamma M_{-\infty}^2 \sin\theta_0 \cdot \Delta_1 + \gamma M_{-\infty}^2 \cos\theta_0 \cdot \Delta_0\theta_1 + \gamma M_{-\infty}^2 \cos\theta_0 \cdot \Delta_1\theta_1 \quad (6a)$$

i.e.,

$$P_2/P_{-\infty} = P_{+\infty}/P_{-\infty} + \gamma M_{-\infty}^2\Delta_1 \cdot \sin\theta_0 + \gamma M_{-\infty}^2 \cos\theta_0 \cdot \theta_1\Delta_0 + \gamma M_{-\infty}^2 \cos\theta_0\theta_1\Delta_1 \quad (6b)$$

where $P_{+\infty}/P_{-\infty} = P_1/P_{-\infty}$ = pressure ratio across the shock for the corresponding inviscid flow.

A. Wedge case

As in Ref. 1, for the case of wedge,

$$\theta_1 \approx [(\gamma + 1)/4]\Delta_1 \quad (7)$$

when θ_0 is also very much less than 1 rad, and $\theta_0 > \theta_1$.

Approximating (6b) in view with (7),

$$\frac{P_2}{P_{+\infty}} = \frac{P_2}{P_1} = 1 + \gamma M_{-\infty}^2 \sin\theta_0\Delta_1 \cdot \frac{P_{-\infty}}{P_{+\infty}} + \gamma M_{-\infty}^2 \cos\theta_0 \frac{P_{-\infty}}{P_{+\infty}}\theta_1\Delta_0 + \gamma M_{-\infty}^2 \cos\theta_0 \frac{P_{-\infty}}{P_{+\infty}}\Delta_1\theta_1$$

i.e.,

$$P^* = 1 + \frac{P_{-\infty}}{P_{+\infty}}\gamma M_{-\infty}^2 \sin\theta_0 \cdot \Delta_1 + \frac{P_{-\infty}}{P_{+\infty}}\gamma M_{-\infty}^2 \cos\theta_0 \cdot \Delta_0 \frac{\gamma + 1}{4}\Delta_1 + \gamma M_{-\infty}^2 \frac{P_{-\infty}}{P_{+\infty}}\cos\theta_0 \cdot \frac{\gamma + 1}{4}\Delta_1^2$$

but

$$\Delta_1 \approx d\eta/d\xi \quad (8)$$